Electromagnetic Waves

30TH NOVEMBER 2020

Maxwell's Equations, General Set

Point Form	Integral Form
$\nabla\cdot\vec{D}=\rho \qquad (1)$ $\nabla\cdot\vec{B}=0 \qquad (2)$ $\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad (3)$ $\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t} \quad (4)$ (1) and (2) are steady state. (3) and (4) are time varying.	$\oint\vec{D}\cdot d\vec{s}=\int\rho dV \qquad (Gauss's\ law\ for\ electric\ field)$ $\oint\vec{B}\cdot d\vec{s}=0 \qquad (Gauss's\ law\ for\ magnetic\ field)$ $\oint\vec{E}\cdot d\vec{l}=\ \int\left(-\frac{\partial\vec{B}}{\partial t}\right)\cdot d\vec{s} \qquad (Faraday's\ law)$ $\oint\vec{H}\cdot d\vec{l}=\ \int\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)\cdot d\vec{s} \qquad (Ampere-Maxwell\ law)$

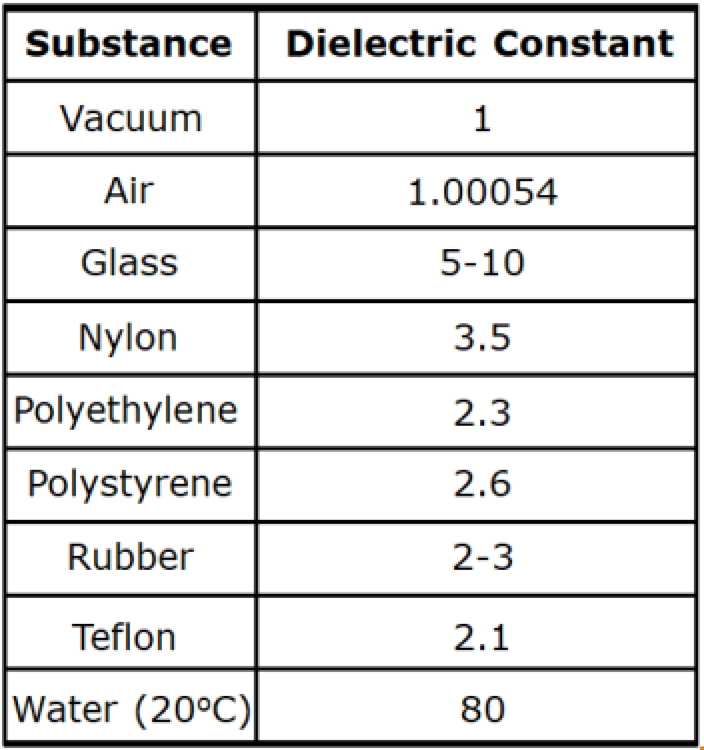
Significance of the Maxwell's equations

- The two time-varying equations are mathematically sufficient to produce separate **wave equations** for the electric and magnetic field vectors.
- Also, they indicates time variable **E** and **H** fields cannot exist independently.
- The steady state equations help to identify the wave nature as **transverse**.
- Two constitutive equations are needed for solving the Maxwell's equations.

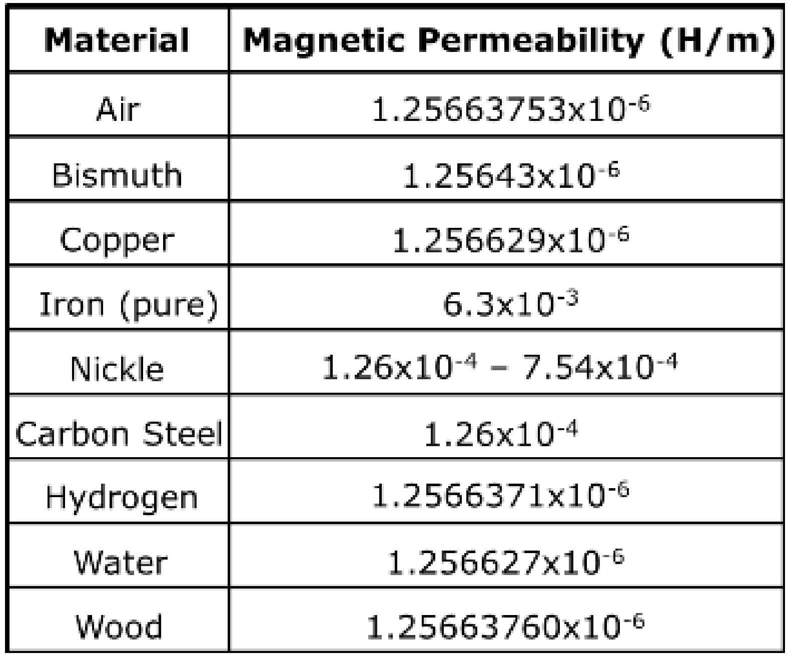
constitutive equaiton; $\vec{D} = \varepsilon\vec{E},\ \vec{B} = \mu\vec{H}$

Permittivity (ε) and permeability (μ)

Permittivity	Permeability
The measure of the **resistance** that is encountered when forming an electric field in the medium.	The measure of the ability of material to **support** the forming of a magnetic field within itself.
Relative permittivity or dielectric constant $\varepsilon_r = {}^{\varepsilon}/_{\varepsilon_0}$	Relative permeability $\mu_r = {}^{\mu}/_{\mu_0}$
The permittivity is relevant to the electric polarization.	The permeability is relevant to the magnetization or magnetic polarization.



Substance	Dielectric Constant
Vacuum	1
Air	1.00054
Glass	5-10
Nylon	3.5
Polyethylene	2.3
Polystyrene	2.6
Rubber	2-3
Teflon	2.1
Water (20°C)	80

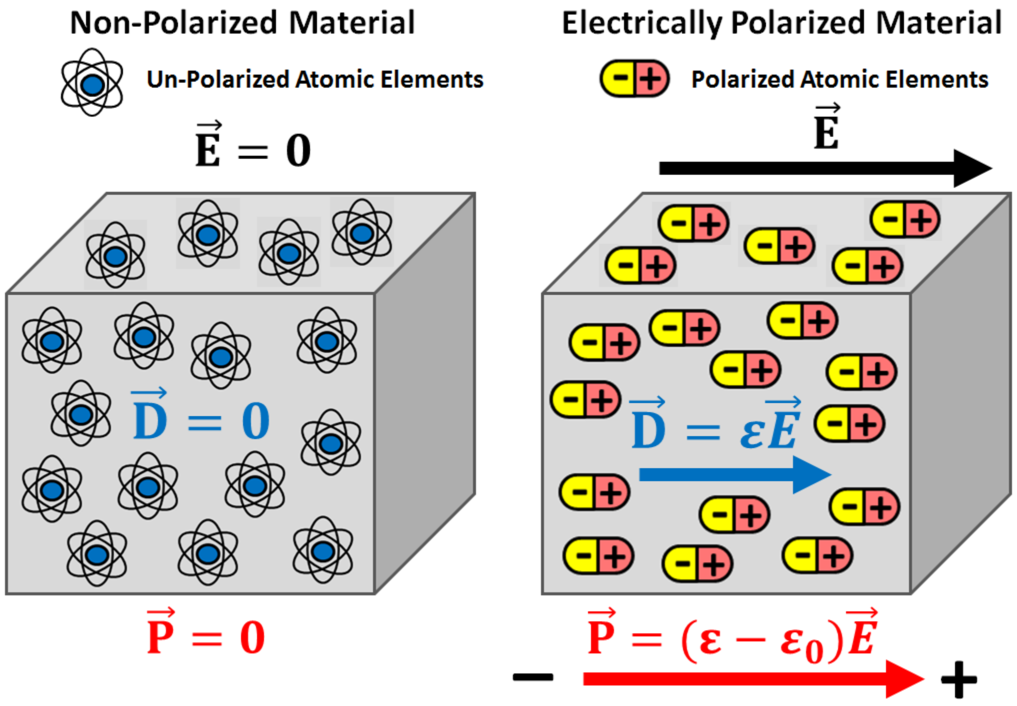


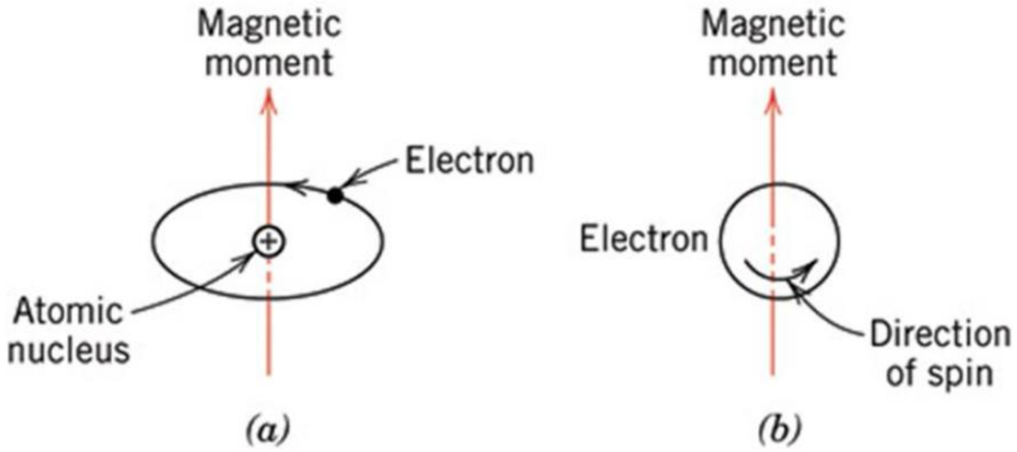
Material	Magnetic Permeability (H/m)
Air	$1.25663753 \times 10^{-6}$
Bismuth	1.25643×10^{-6}
Copper	1.256629×10^{-6}
Iron (pure)	6.3×10^{-3}
Nickle	1.26×10^{-4} – 7.54×10^{-4}
Carbon Steel	1.26×10^{-4}
Hydrogen	1.2566371×10^{-6}
Water	1.256627×10^{-6}
Wood	$1.25663760 \times 10^{-6}$

https://study.com/academy/lesson/what-is-permittivity-definition-examples.html

Origin of polarization

Origin of magnitization





https://em.geosci.xyz/_images/electric_polarization_physics_diagram.png

https://slideplayer.com/slide/8700248/

Scopes of study

- Electromagnetic waves in a medium having finite permeability μ and permittivity ε but with conductivity $\sigma = 0$,
- The wave equation for electromagnetic Waves in a dielectric,
- Impedance of a dielectric to electromagnetic Waves,
- Electromagnetic waves in a medium of properties μ, ε and σ (where $\sigma \neq 0$),
- Electromagnetic wave velocity in a conductor and anomalous dispersion,

Electromagnetic waves in a medium having finite **permeability μ** and **permittivity ε** but with **conductivity σ = 0**

- Given conditions

(1) The properties (eg. amplitude, phase, frequency, wavelength and speed) of the chosen plane waves in *xy* plane are constant,

(2) These properties will not vary with respect to *x* and *y* and all derivatives $\partial/\partial x$ and $\partial/\partial y$ will be zero.

(3) In dielectric such as free space, no charge ($\rho = 0$) and no current density ($J = 0$).

Maxwell’s equations in dielectric

$$\nabla \cdot \vec{D} = \rho \qquad (1)$$

$$\nabla \cdot \vec{B} = 0 \qquad (2)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad (3)$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad (4)$$

$$\varepsilon\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = 0$$

(1A)

$$\mu\left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\right) = 0$$

(2A)

$$\left.\begin{aligned} -\mu\frac{\partial}{\partial t}H_x &= \frac{\partial}{\partial y}E_z - \frac{\partial}{\partial z}E_y \\ -\mu\frac{\partial}{\partial t}H_y &= \frac{\partial}{\partial z}E_x - \frac{\partial}{\partial x}E_z \\ -\mu\frac{\partial}{\partial t}H_z &= \frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}E_x \end{aligned}\right\}$$

(3A)

$$\left.\begin{aligned} \varepsilon\frac{\partial}{\partial t}E_x &= \frac{\partial}{\partial y}H_z - \frac{\partial}{\partial z}H_y \\ \varepsilon\frac{\partial}{\partial t}E_y &= \frac{\partial}{\partial z}H_x - \frac{\partial}{\partial x}H_z \\ \varepsilon\frac{\partial}{\partial t}E_z &= \frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x \end{aligned}\right\}$$

(4A)

The wave equation of electromagnetic waves in a dielectric

- Since, with these plane waves, all derivatives with respect to x and y are zero. Equations (2A) and (3A) give

$$\frac{\partial H_z}{\partial z} = 0 \quad \text{and} \quad -\mu\frac{\partial H_z}{\partial t} = 0$$

- This implies that H_z is constant in space and time.
- H_z has no effect on the wave motion. For simplicity, put $H_z = 0$.
- A similar consideration of equations (1A) and (4A) leads to $E_z = 0$.
- The above results suggest that the oscillations in ***H* and *E* occur in direction perpendicular to *z*-direction**. In other words, the **EM plane wave is transverse**.

The wave equation for electromagnetic waves in a dielectric : **plane-polarized waves (1)**

- Consider E_x only, with $E_y = 0$, equations (3A) and (4A) give

$$-\mu\frac{\partial H_y}{\partial t} = \frac{\partial E_x}{\partial z} \qquad \varepsilon\frac{\partial E_x}{\partial t} = -\frac{\partial H_y}{\partial z}$$

- Using the fact that

$$\frac{\partial^2}{\partial z \partial t} = \frac{\partial^2}{\partial t \partial z}$$

- The wave equation for H_y is found to be

$$\boxed{\frac{\partial^2}{\partial z^2}H_y = \mu\varepsilon\frac{\partial^2}{\partial t^2}H_y}$$

- Similarly, the wave equation for E_x is

$$\boxed{\frac{\partial^2}{\partial z^2}E_x = \mu\varepsilon\frac{\partial^2}{\partial t^2}E_x}$$

Recall a formal derivation of the wave equation from Maxwell's equations in dielectric (1)

- From identity $\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E})\text{-}\nabla^2 \vec{E}$
- From the constitutive equation :$\nabla \times \nabla \times \vec{E} = \frac{1}{\varepsilon}\nabla(\nabla \cdot \vec{D}) - \nabla^2 \vec{E}$
- Due to equation (1) of Maxwell's equation: $\nabla \times \nabla \times \vec{E} = -\nabla^2 \vec{E}$ (for dielectric ρ = 0)
- From equation (3) of Maxwell's equation: $$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\nabla^2 \vec{E}$$
- From the constitutive equation : $$\nabla \times \left(-\frac{\partial \mu \vec{H}}{\partial t} \right) = -\frac{\partial}{\partial t} \mu \left(\nabla \times \vec{H} \right) = -\nabla^2 \vec{E}$$
- From equation (4) and constitutive equation : $$-\frac{\partial}{\partial t} \mu \left(\frac{\partial \vec{D}}{\partial t} \right) = -\mu\varepsilon \frac{\partial^2}{\partial t^2} \vec{E} = -\nabla^2 \vec{E}$$

Recall a formal derivation of the wave equation from Maxwell's equations in dielectric (2)

- The wave equation $\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2}{\partial t^2} \vec{E}$
- Possible solution for the wave equation may be given as $\vec{E}(z,t) = \vec{E}_0 e^{i(\omega t - kz)}$
- By substituting the solution in the wave equation, we obtain

$$\nabla^2 \vec{E} = -k^2 \vec{E}_0 e^{i(\omega t - kz)} = -\mu\varepsilon\omega^2 \vec{E}_0 e^{i(\omega t - kz)}$$

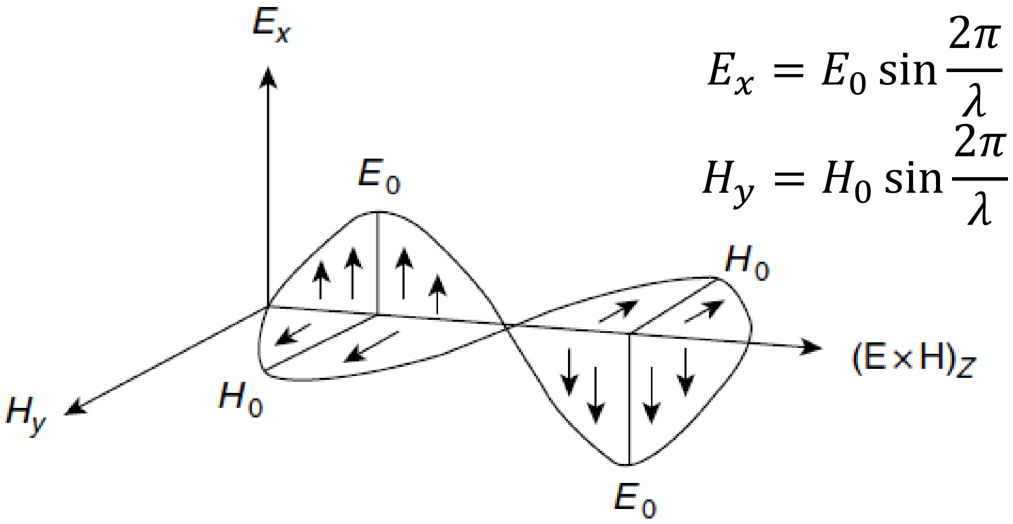
$$\therefore k = \omega\sqrt{\mu\varepsilon} \Rightarrow \frac{k}{\omega} = \frac{1}{v} = \sqrt{\mu\varepsilon}$$

$$\therefore \frac{1}{c^2} = \mu_0\varepsilon_0 \ \left(\text{in free space}\right)$$

- Finally, the wave equation in free space becomes $\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E}$

The wave equation for electromagnetic Waves in dielectric : **plane-polarized waves (2)**

- The vector E_x and H_y obey the same wave equation.
- In free space, the velocity is that of light given as $c^2 = 1/\mu_0\varepsilon_0$



Solutions of the wave equations

$$E_x = E_0 \sin \frac{2\pi}{\lambda}(vt - z)$$

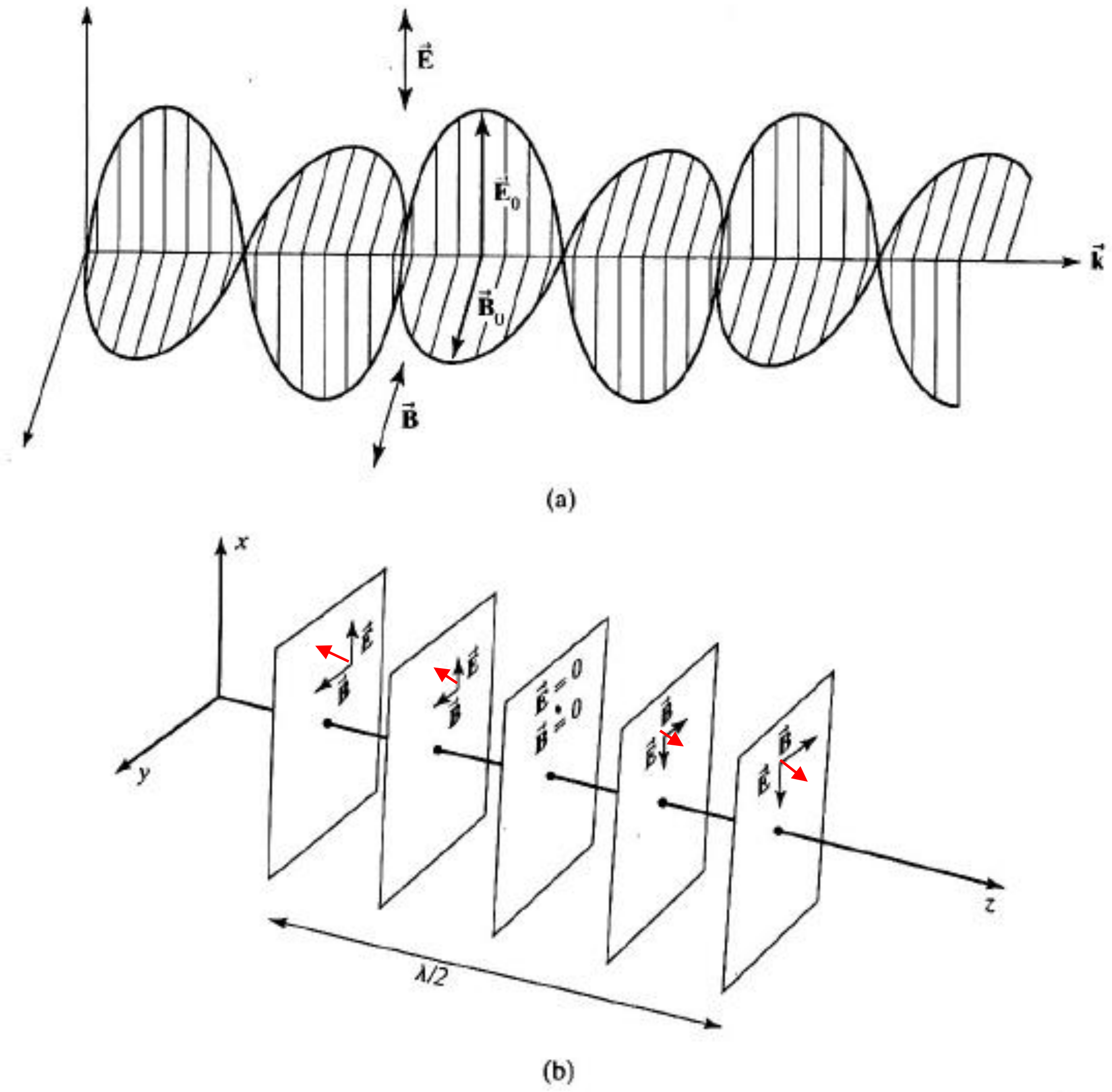
$$H_y = H_0 \sin \frac{2\pi}{\lambda}(vt - z)$$

The vector product gives direction of the energy flow. This can be seen from the product dimension:

$$\frac{\text{voltage} \times \text{current}}{\text{length} \times \text{length}} = \frac{\text{electrical power}}{\text{area}}$$

Figure 8.3 In a plane-polarized electromagnetic wave the electric field vector E_x and magnetic field vector H_y are perpendicular to each other and vary sinusoidally. In a non-conducting medium they are in phase. The vector product, $\mathbf{E} \times \mathbf{H}$, gives the direction of energy flow

Plane electromagnetic waves



(a) The electric field E, magnetic field B and propagation vector k are everywhere mutually perpendicular

(b) Wave fronts for a linear polarized plane electromagnetic wave

F Pedrotti et. al. “Introduction to optics”, 3ed., Pearson international (2007)

Energy density of electromagnetic wave in free space

- An electromagnetic wave represents the transmission of energy.
- The energy density, in J/m^3 , for electric field u_E and magnetic field u_B in free space are given as

$$u_E = \frac{1}{2}\varepsilon_0 E^2,\ u_B = \frac{1}{2}\frac{1}{\mu_0}B^2$$

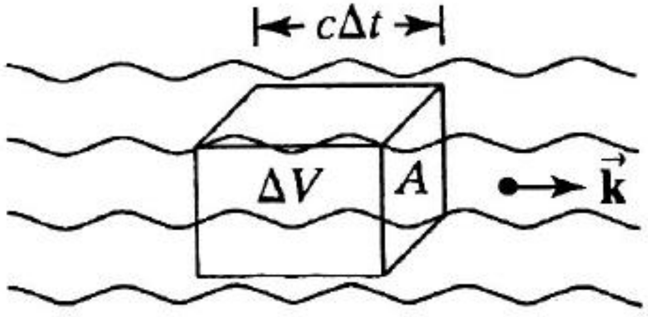
- At any specified time and place, the two field are related by $E = cB$.
- This gives $u_E = u_B$ and the total energy density is $u = u_E + u_B = 2u_E = 2u_B$
- Or

$$u = \varepsilon_0 E^2 = \frac{1}{\mu_0}B^2$$

Poynting vector (1)

- Consider the power or the rate at which energy is transported by the electromagnetic wave.
- Energy flow of an EM wave in time Δt. The energy enclosed in the rectangular volume ΔV flows across the surface A.

$$power = \frac{energy}{\Delta t} = \frac{u\Delta V}{\Delta t} = \frac{u\left(Ac\Delta t\right)}{\Delta t} = ucA$$



- Therefore, power transferred per unit area S is

$$\frac{\text{Power}}{A} = S = uc$$

Poynting vector (2)

- The power per unit area in terms of *E* and *B* is given as

$$S = \varepsilon_0 c^2 EB$$

- The power unit area, *S*, when assigned the direction of propagation, is called **Poynting vector**. This can be written as

$$\vec{S} = \varepsilon_0 c^2 \vec{E} \times \vec{B}$$

- Time **average of the power per unit area** called **irradiance** is given as

$$I = \left\langle \left| \vec{S} \right| \right\rangle$$

Problem : Poynting vector (energy flow in w/m^2)

The plane polarized electromagnetic wave (E_x, H_y) travels in free space.

Show that its Poynting vector is given by

$$S = E_x H_y = c\varepsilon_0 E_x^2$$

Where c is the velocity of light.

Determine the intensity in such a wave.

Solution

- Due to $\vec{S} = \varepsilon_0 c^2 \vec{E} \times \vec{B}$ and the constitutive equation $\vec{B} = \mu_0 \vec{H}$
- The Poynting vector becomes $\vec{S} = \vec{E} \times \vec{H}$
- With appropriate solution for the plane wave; $E_x = E_0 \sin \frac{2\pi}{\lambda}(vt - z)$

$$H_y = H_0 \sin \frac{2\pi}{\lambda}(vt - z)$$

- This gives $\vec{S} = E_x H_y \hat{k}$
- Using equation (3) from page 8 ;$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}$, this leads to $-\mu \frac{\partial H_y}{\partial t} = \frac{\partial E_x}{\partial z}$
- Therefore, $\frac{E_0}{H_0} = \frac{E_x}{H_y} = \sqrt{\frac{\mu}{\varepsilon}}$

Solution

$\because$ in free space; $H_y = E_x \sqrt{\dfrac{\varepsilon_0}{\mu_0}} \Rightarrow S = E_x H_y = c\varepsilon_0 E_x^2$

- The intensity of this wave can be calculated from $I = \left\langle \left| \vec{S} \right| \right\rangle$

$$I = \left\langle \left| \vec{S} \right| \right\rangle = c\varepsilon_0 E_0^2 \left\langle \sin^2 \frac{2\pi}{\lambda}(vt - z) \right\rangle$$

$$= \frac{1}{2} c\varepsilon_0 E_0^2$$

Problem : average irradiance

A laser beam of radius 1 mm carries a power of 6 kW.

Determine its average irradiance and the amplitude of its *E* and *B* fields.

Solution

- The average irradiance can be found from power/area = 1.91×10^9 W/m^2

- From the irradiance $I = \frac{1}{2} c\varepsilon_0 E_0^2$, the amplitude of E field is found to be 1.20×10^6 V/m

- Due to the relationship $E = cB$, the amplitude of B field becomes 4.00×10^{-3} T

Impedance of a dielectric to electromagnetic Waves

- From $Z = \frac{E_0}{H_0} = \frac{E_x}{H_y} = \sqrt{\frac{\mu}{\varepsilon}}$

- If the medium is free space and $\mu = \mu_0 = 4\pi \times 10^{-7}\ V \cdot s/A \cdot m$

  $\varepsilon = \varepsilon_0 = 8.8542 \times 10^{-12}\ F/m$

- Also the dimensions of $\sqrt{\frac{\mu_0}{\varepsilon_0}}$ is ohms **(check this!)**

- Therefore, $\sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7\ \Omega$ represents the **free space characteristic impedance**

to electromagnetic wave travelling through it.

Electromagnetic waves in a medium of properties μ, ε and σ (where σ ≠ 0) (1)

- To derive the wave equation of the plane polarized wave composed of E_x and H_y components propagating in **a conductor**, let's start with equation (4) from page 8 with a **current density** term $\vec{J}\left(=\sigma\vec{E}\right)$

$$\nabla\times\vec{H}=\frac{\partial}{\partial t}\vec{D}+\vec{J}$$

- This becomes $$\varepsilon\frac{\partial E_x}{\partial t}+\sigma E_x=-\frac{\partial H_y}{\partial z}$$

- Taking ∂/∂t, $$\varepsilon\frac{\partial^2}{\partial t^2}E_x+\frac{\partial}{\partial t}\sigma E_x=-\frac{\partial^2}{\partial t\partial z}H_y$$

Electromagnetic waves in a medium of properties μ, ε and σ (where σ ≠ 0) (2)

- Also consider equation (3) from page 8 : $\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$

- This gives

$$-\mu \frac{\partial H_y}{\partial t} = \frac{\partial E_x}{\partial z}$$

- Taking ∂/∂z,

$$-\mu \frac{\partial^2}{\partial t \partial z} H_y = \frac{\partial^2}{\partial^2 z} E_x$$

- The final equation as the wave equation for EM waves in a conductor is found to be

$$\boxed{\frac{\partial^2}{\partial z^2} E_x = \mu\varepsilon \frac{\partial^2}{\partial t^2} E_x + \boxed{\mu\sigma \frac{\partial}{\partial t} E_x}} \longleftarrow \text{Diffusion term}$$

Electromagnetic wave in a conductor

- Recall the wave equation of EM plane polarized wave in a conductor,

$$\boxed{\frac{\partial^2}{\partial z^2}E_x = \mu\varepsilon\frac{\partial^2}{\partial t^2}E_x + \mu\sigma\frac{\partial}{\partial t}E_x}$$

- A solution of the wave equation is found to be

$$E_x = E_0\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{-\gamma z}$$

- Where $\gamma^2 = \mathrm{i}\omega\mu\sigma - \omega^2\mu\varepsilon.$

Solution of the wave equation with diffusion effect

- Recall, the wave equation in a conductor,

$$\boxed{\frac{\partial^2}{\partial z^2}E_x = \mu\varepsilon\frac{\partial^2}{\partial t^2}E_x + \mu\sigma\frac{\partial}{\partial t}E_x}$$

- With the assumption that its time-variation is simple harmonic, $E_x = E_0 e^{i\omega t}$

- Substitute the assumed solution into the wave equation,

$$\frac{\partial^2 E_x}{\partial z^2} - (\underbrace{\mathrm{i}\omega\mu\sigma - \omega^2\mu\varepsilon}_{\gamma^2})E_x = 0$$

$$\frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0$$

- Solution of the above 2nd order differential equation is given as

$$E_x = Ae^{\gamma z} + Be^{-\gamma z}$$

- Since the wave propagates in +z direction the second term of the solution is chosen.
- The final solution for the time dependent wave equation in a conductor can be written as

$$E_x = E_0 \, \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{e}^{-\gamma z}$$

- The ratio of the current density terms can be written as

$$\frac{\mathbf{J}}{\partial\mathbf{D}/\partial t}=\frac{\sigma E_x}{\partial/\partial t(\varepsilon E_x)}=\frac{\sigma E_x}{\partial/\partial t(\varepsilon E_0\,\mathrm{e}^{\mathrm{i}\omega t})}=\frac{\sigma E_x}{\mathrm{i}\omega\varepsilon E_x}=\frac{\sigma}{\mathrm{i}\omega\varepsilon}$$

- For a conductor, where $\vec{J} \gg \partial\vec{D}/\partial t$, then $\sigma \gg \omega\varepsilon \rightarrow \gamma^2 = i\sigma(\omega\mu)\text{-}\omega\varepsilon\ (\omega\mu)$

- This can be written as

$$\gamma^2 \approx \mathrm{i}\sigma\omega\mu$$

$$\Rightarrow \quad \gamma = (1+\mathrm{i})\left(\frac{\omega\mu\sigma}{2}\right)^{1/2}$$

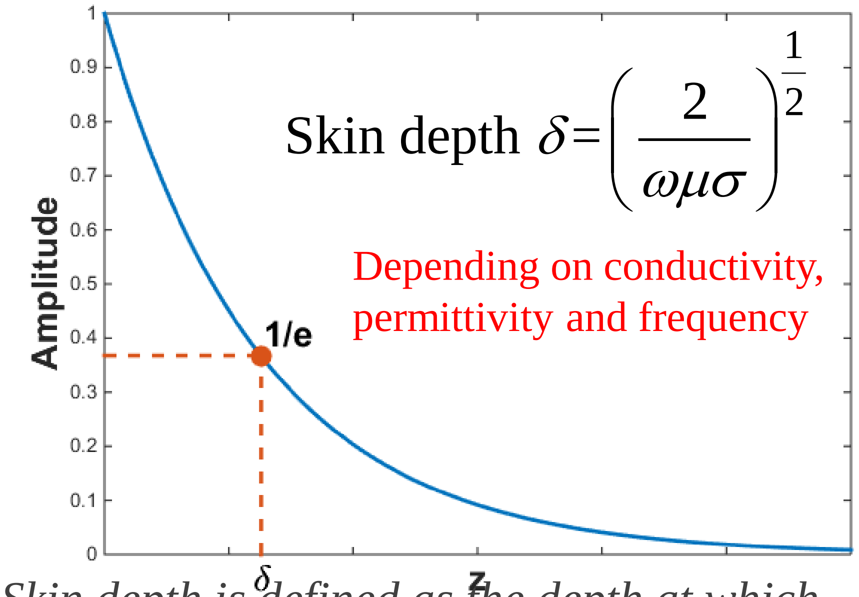
- The wave function becomes

$$\begin{aligned} E_x &= E_0\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{-\gamma z} \\ &= E_0\,\boxed{\mathrm{e}^{-(\omega\mu\sigma/2)^{1/2}z}}\,\mathrm{e}^{\mathrm{i}[\omega t-(\omega\mu\sigma/2)^{1/2}z]} \end{aligned}$$

Exponential decay term for
the amplitude as a function of distance z

Skin depth

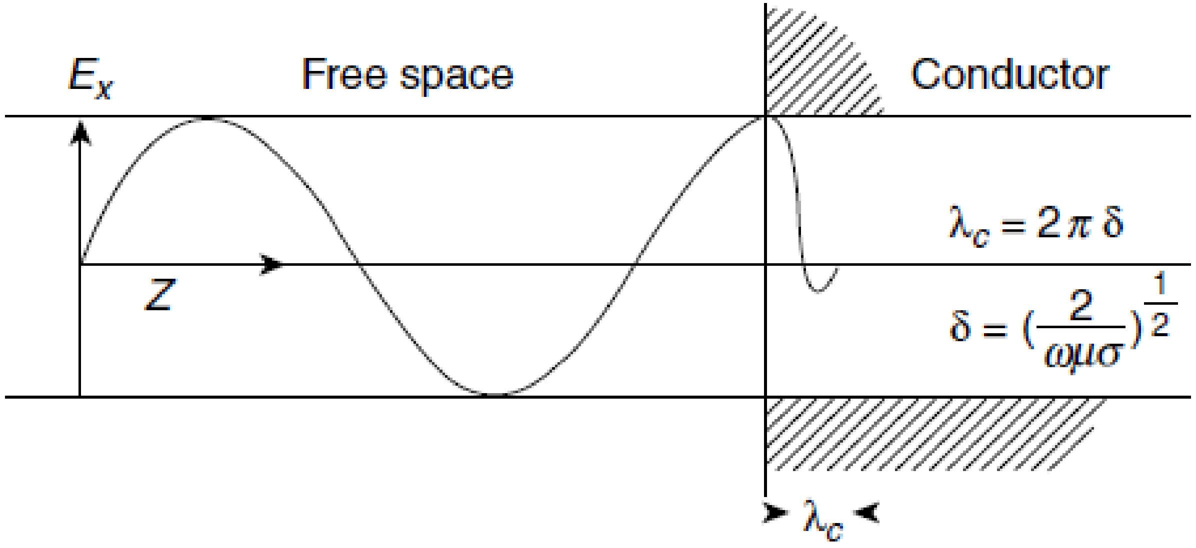
- The skin depth is the travelling distance of EM wave in the conductor when the electric field vector has decayed to a value of $E_x = E_0 e^{-1}$.



$$\text{Skin depth } \delta = \left(\frac{2}{\omega\mu\sigma} \right)^{\frac{1}{2}}$$

Depending on conductivity, permittivity and frequency

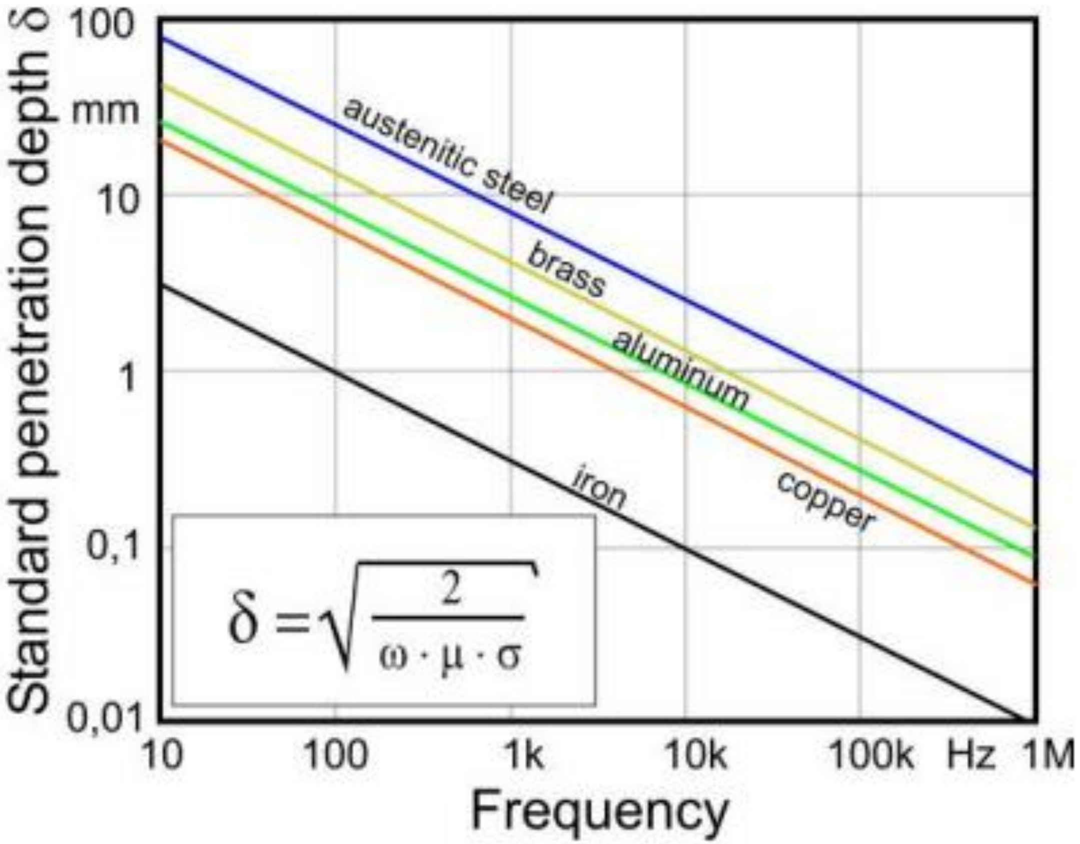
Skin depth is defined as the depth at which the amplitude of the wave has been reduced by 1/e



This explains the electrical shielding properties of a conductor, λ_c **is the wavelength in the conductor**.

https://em.geosci.xyz/content/maxwell1_fundamentals/harmonic_planewaves_homogeneous/skindepth.html

Example of skin depth for metals



- Note that as the frequency of the EM wave increases, the penetration depth decreases.
- This means the EM wave hardly penetrates into the conducting material.

https://www.ndt.net/article/ecndt2006/doc/Tu.3.6.2.pdf

Electromagnetic wave velocity in a conductor and anomalous dispersion

- Recall the wave function in a conductor

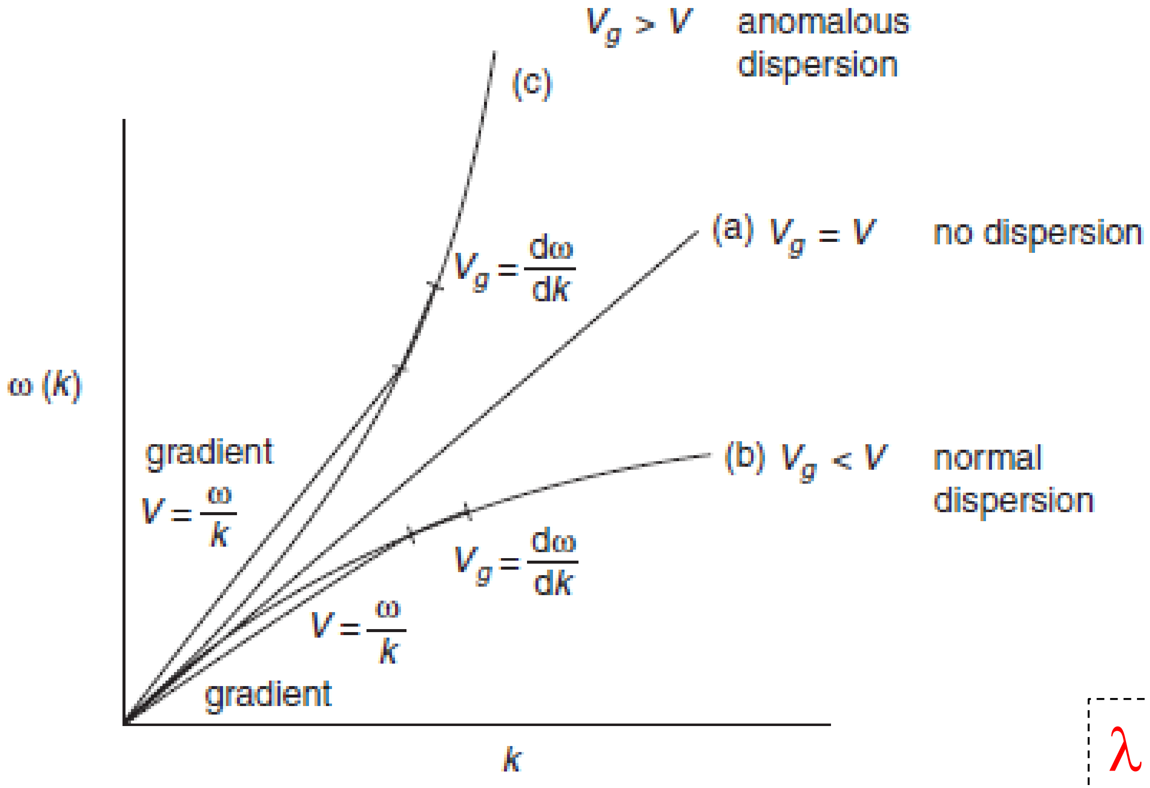
$$E_x = E_0 \mathrm{e}^{\mathrm{i}\omega t} \mathrm{e}^{-\gamma z} = E_0 \mathrm{e}^{-(\omega\mu\sigma/2)^{1/2} z} \mathrm{e}^{\mathrm{i}[\omega t-(\omega\mu\sigma/2)^{1/2} z]}$$

- The **phase velocity** of the wave υ is given by

$$v = \frac{\omega}{k} = \frac{\omega}{(\omega\mu\sigma/2)^{1/2}} = \omega\delta = \left(\frac{2\omega}{\mu\sigma}\right)^{1/2} = \nu\lambda_c \quad ;(\lambda_c = 2\pi\delta)$$

- Since the phase velocity **is a function of the frequency**, an electrical conductor is **dispersive medium** to EM waves.
- Because $\partial v/\partial\lambda$ is negative so that the conductor is anomalously dispersive and the group velocity is greater than the phase velocity.

Recall the illustration of dispersion relation



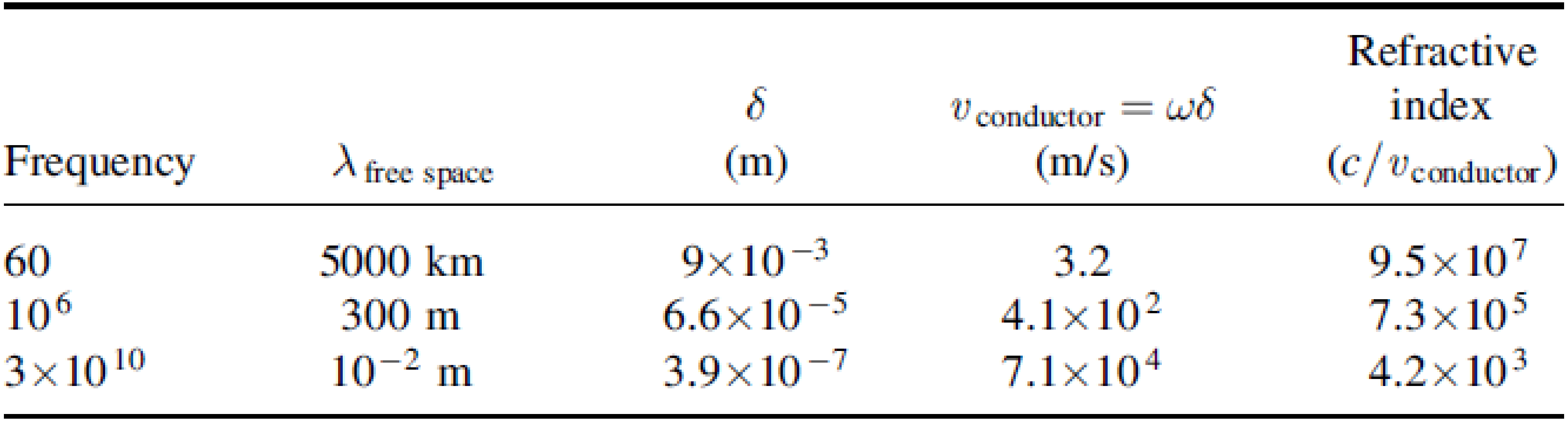
- Since $\omega = kv$,

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}\left(kv\right) = v + k\frac{dv}{dk}$$

$$= v - \lambda \boxed{\frac{dv}{d\lambda}}$$

This term can be either positive or negative.

λ ↑ v ↑ : $v_g < v$ (normal dispersion)
λ ↑ v ↓ : $v_g > v$ (anomalous dispersion)



Frequency	$\lambda_{\text{free space}}$	δ (m)	$v_{\text{conductor}} = \omega\delta$ (m/s)	Refractive index ($c/v_{\text{conductor}}$)
60	5000 km	9×10^{-3}	3.2	9.5×10^{7}
10^{6}	300 m	6.6×10^{-5}	4.1×10^{2}	7.3×10^{5}
3×10^{10}	10^{-2} m	3.9×10^{-7}	7.1×10^{4}	4.2×10^{3}

- High frequency EM waves propagate only a very small distance in conductor.
- When δ is small, the phase velocity v is small, and the refractive index c/v of a conductor can be very large. This results in the high optical reflectivity.

Determine the distance in which the amplitude of the striking EM wave drops to about 1% of its surface value.

Solution

- Recall the electric field in the conductor

$$E_x = E_0\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{e}^{-\gamma z}$$

$$= E_0\, \mathrm{e}^{-(\omega\mu\sigma/2)^{1/2} z}\, \mathrm{e}^{\mathrm{i}[\omega t - (\omega\mu\sigma/2)^{1/2} z]}$$

- At the surface of the conductor, the amplitude of the incident wave is given by E_0.
- The amplitude reduced to 1% of its surface value is given by $0.01E_0e^{-(\omega\mu\sigma/2)^{\frac{1}{2}}z}$ or $0.01E_0e^{-\frac{z}{\delta}}$.
- This leads to the travelling distance of 4.6δ.

When is a medium a conductor or a dielectric?

- The ratio of the current density is used to determine whether the medium is a conductor or a dielectric.

$$\frac{J}{\partial D/\partial t} = \frac{\sigma}{\omega\varepsilon}$$

- The conduction current dominates and the medium is a conductor :

$$\frac{J}{\partial D/\partial t} = \frac{\sigma}{\omega\varepsilon} > 100$$

- The displacement current dominates and the material behaves as a dielectric :

$$\frac{\partial D/\partial t}{J} = \frac{\omega\varepsilon}{\sigma} > 100$$

Problem 8.14

A medium has a conductivity $\sigma = 10^{-1}$ S m^{-1} and a relative permittivity $\varepsilon_r = 50$, which is constant with frequency. If the relative permeability $\mu_r = 1$, is the medium a conductor or a dielectric at a frequency of (a) 50 kHz, and (b) 10^4 MHz?

$$[\varepsilon_0 = (36\pi \times 10^9)^{-1}\,\mathrm{F\,m^{-1}};\ \mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}]$$

Problem 8.15

The electrical properties of the Atlantic Ocean are given by

$$\varepsilon_r = 81, \quad \mu_r = 1, \quad \sigma = 4.3\,\mathrm{S\,m^{-1}}$$

Show that it is a conductor up to a frequency of about 10 MHz. What is the longest electromagnetic wavelength you would expect to propagate under water?

Impedance of a conduction medium to EM waves

- Recall the electric field in a conductor $E_x = E_0 e^{i\omega t} e^{-\gamma z}$
- The corresponding magnetic filed in the conductor is given as $H_y = H_0 e^{i(\omega t - \phi)} e^{-\gamma z}$
- Given that $\gamma = (1+i)(\omega\mu\sigma/2)^{\frac{1}{2}}$
- The impedance of the conductor is given as $Z_c = \dfrac{E_x}{H_y}$
- By using time-dependent Maxwell's equation $\nabla \times \vec{E} = -\mu \dfrac{\partial}{\partial t} \vec{H}$

$$Z_c = \frac{E_x}{H_y} = \frac{i\omega\mu}{\gamma} = \left(\frac{\omega\mu}{\sigma}\right)^{\frac{1}{2}} \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \left(\frac{\omega\mu}{\sigma}\right)^{\frac{1}{2}} e^{i\left(\frac{\pi}{4}\right)}$$

$$Note: \sqrt{i} = \frac{1+i}{\sqrt{2}}$$

The magnitude of Z_C

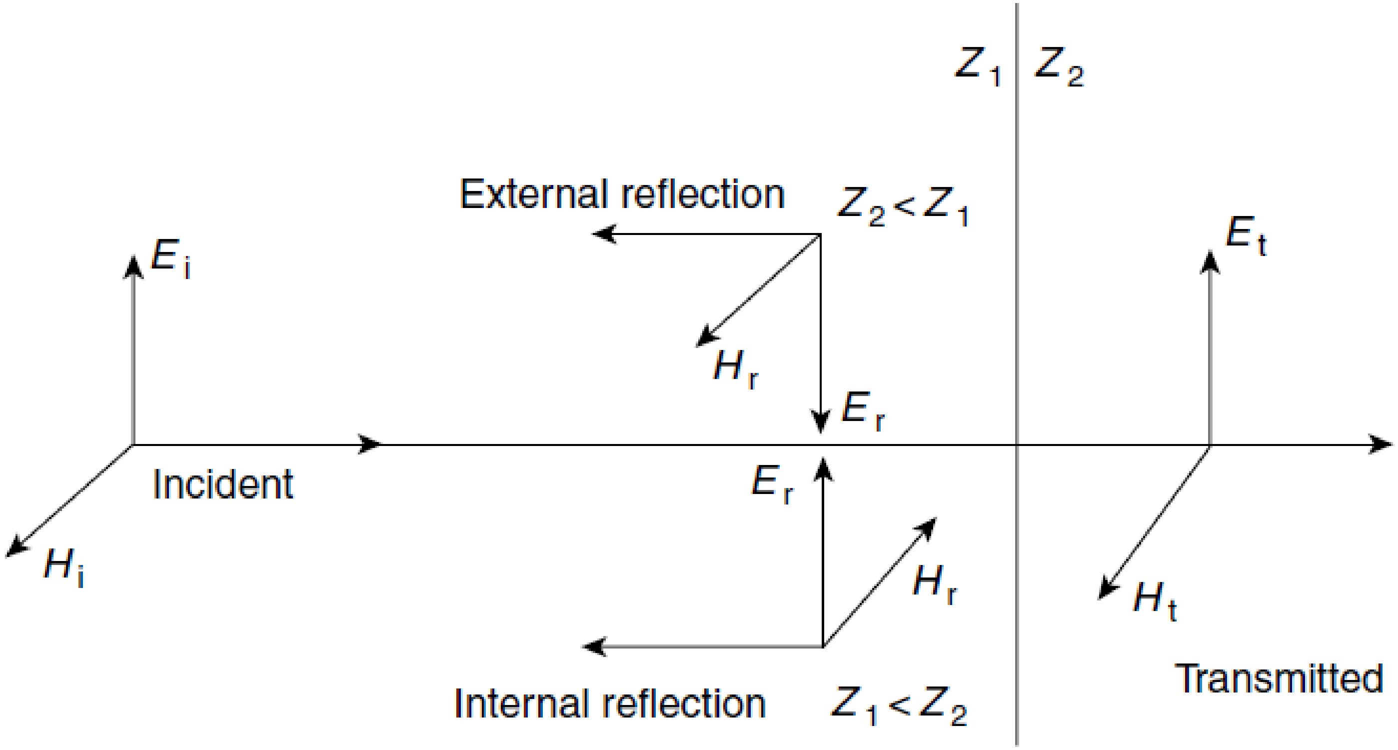
- Recall the impedance of a conducting medium,

$$Z_c = \left(\frac{\omega\mu}{\sigma}\right)^{\frac{1}{2}} e^{i\left(\frac{\pi}{4}\right)}$$

- The magnitude can be written in terms of the free space impedance as follows,

$$\left|Z_c\right| = 376.6\Omega\sqrt{\frac{\mu_r}{\varepsilon_r}}\sqrt{\frac{\omega\varepsilon}{\sigma}}$$

Reflection and transmission of EM waves at a boundary : Normal incidence



$$E_i + E_r = E_t$$

$$H_i + H_r = H_t$$

$$\frac{E_i}{H_i} = Z_1; \frac{E_r}{H_r} = -Z_1; \frac{E_t}{H_t} = Z_2$$

The boundary conditions, from EM theory, are that the components of **the field vectors E and H tangential or parallel to the boundary are continuous across the boundary**.

Reflection and transmission coefficients : normal incidence

$$R = \frac{E_r}{E_i} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$T = \frac{E_t}{E_i} = \frac{2Z_2}{Z_2 + Z_1}$$

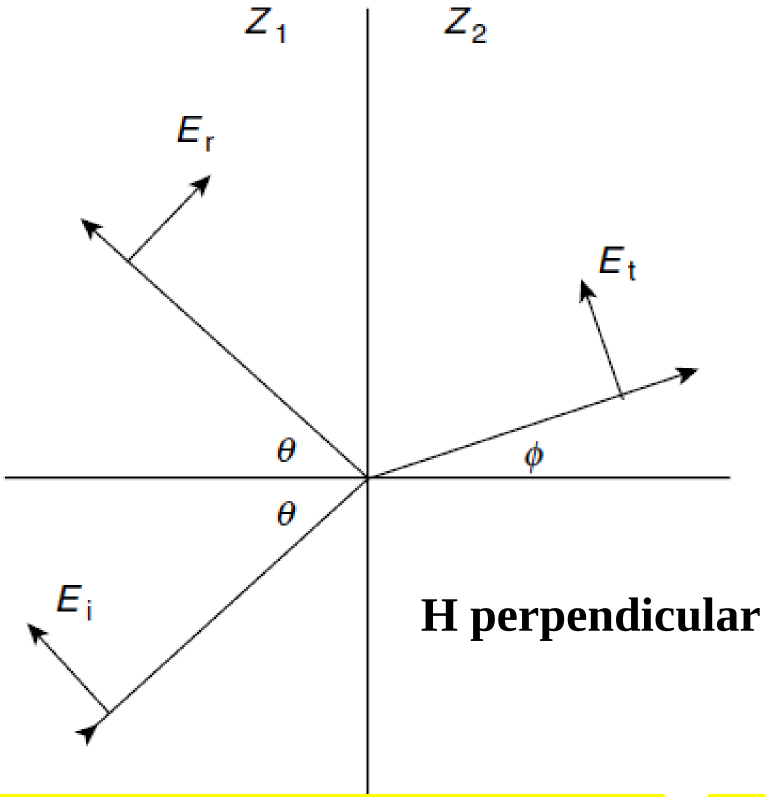
- **What happens to the travelling wave normally striking a perfect conductor in terms of reflection and transmission coefficients?**

Oblique incidence and Fresnel’s Equations for dielectrics

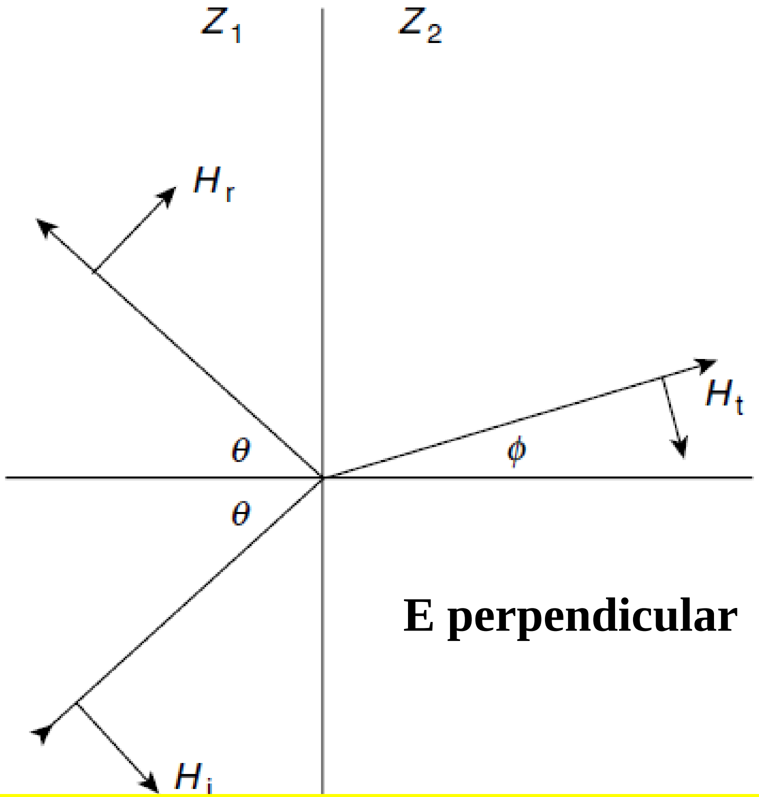
- The same boundary conditions are still applied.
- Two cases have to be investigated :

(1) **H** perpendicular to the plane of incidence and

(2) **E** perpendicular to the plane of incidence



H perpendicular



E perpendicular

$$H_i + H_r = H_t$$
$$E_i \cos\theta + E_r \cos\theta = E_t \cos\phi$$

$$E_i + E_r = E_t$$
$$-H_i \cos\theta + H_r \cos\theta = -H_t \cos\phi$$

Verify the direction of H_i in the figure.

Reflection and transmission coefficients : oblique incidence

For H perpendicular to the plane of incidence

$$R_{\parallel} = \frac{E_r}{E_i} = \frac{Z_2 \cos\phi - Z_1 \cos\theta}{Z_2 \cos\phi + Z_1 \cos\theta}$$

$$T_{\parallel} = \frac{E_t}{E_i} = \frac{2Z_2 \cos\theta}{Z_2 \cos\phi + Z_1 \cos\theta}$$

For E perpendicular to the plane of incidence

$$R_{\perp} = \frac{Z_2 \cos\theta - Z_1 \cos\phi}{Z_2 \cos\theta + Z_1 \cos\phi}$$

$$T_{\perp} = \frac{2Z_2 \cos\theta}{Z_2 \cos\theta + Z_1 \cos\phi}$$

Reflection and transmission coefficients in terms of refractive index *n*

- Since $$n = \frac{c}{v} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = \sqrt{\varepsilon_r} = \frac{Z\left(\text{free space}\right)}{Z\left(\text{dielectric}\right)}$$

- The reflection and transmission coefficients can be expressed in terms of refractive indices of incident (n_1) and transmitted (n_2)media.

$$R_{\parallel} = \frac{E_r}{E_i} = \frac{n_1 \cos\phi - n_2 \cos\theta}{n_1 \cos\phi + n_2 \cos\theta}$$

$$R_{\perp} = \frac{n_1 \cos\theta - n_2 \cos\phi}{n_1 \cos\theta + n_2 \cos\phi}$$

$$T_{\parallel} = \frac{E_t}{E_i} = \frac{2n_1 \cos\theta}{n_1 \cos\phi + n_2 \cos\theta}$$

$$T_{\perp} = \frac{2n_1 \cos\theta}{n_1 \cos\theta + n_2 \cos\phi}$$

Fresnel's equations

$$R_{\parallel} = \frac{\tan\,(\phi - \theta)}{\tan\,(\phi + \theta)}, \qquad T_{\parallel} = \frac{4\,\sin\,\phi\,\cos\,\theta}{\sin\,2\phi + \sin\,2\theta}$$

$$R_{\perp} = \frac{\sin\,(\phi - \theta)}{\sin\,(\phi + \theta)}, \qquad T_{\perp} = \frac{2\,\sin\,\phi\,\cos\,\theta}{\sin\,(\phi + \theta)}$$

What are the reflection coefficients at the normal incidence in terms of refractive indices?

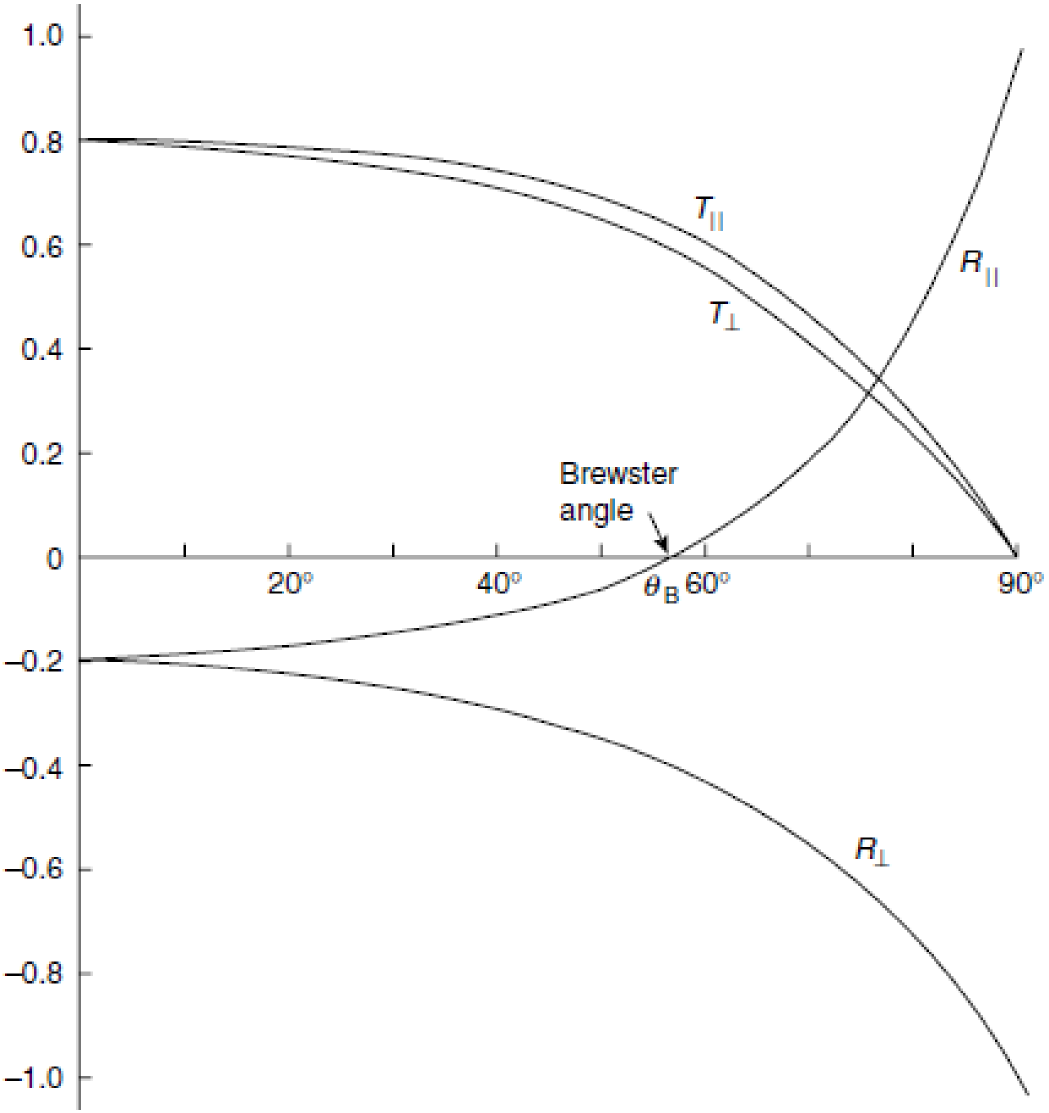


Figure 8.9 Amplitude coefficient R and T of reflection and transmission for $n_2/n_1 = 1.5$. $R_{\parallel}$ and $T_{\parallel}$ refer to the case when the electric field vector E lies in the plane of incidence. $R_{\perp}$ and $T_{\perp}$ apply when E is perpendicular to the plane of incidence. The Brewster angle θ_B defines $\theta + \phi = 90°$ when $R_{\parallel} = 0$ and the reflected light is polarized with the E vector perpendicular to the plane of incidence. $R_{\parallel}$ changes sign (phase) at θ_B. When $\theta < \theta_B$, tan $(\phi - \theta)$ is negative for $n_2/n_1 = 1.5$. When $\theta + \phi \geq 90°$, tan $(\phi + \theta)$ is also negative

Brewster angle or polarizing angle θ_B can be found from

$$\tan\theta_B = n_2/n_1$$

Recall : $R_{\parallel} = \frac{\tan(\theta - \phi)}{\tan(\theta + \phi)}$

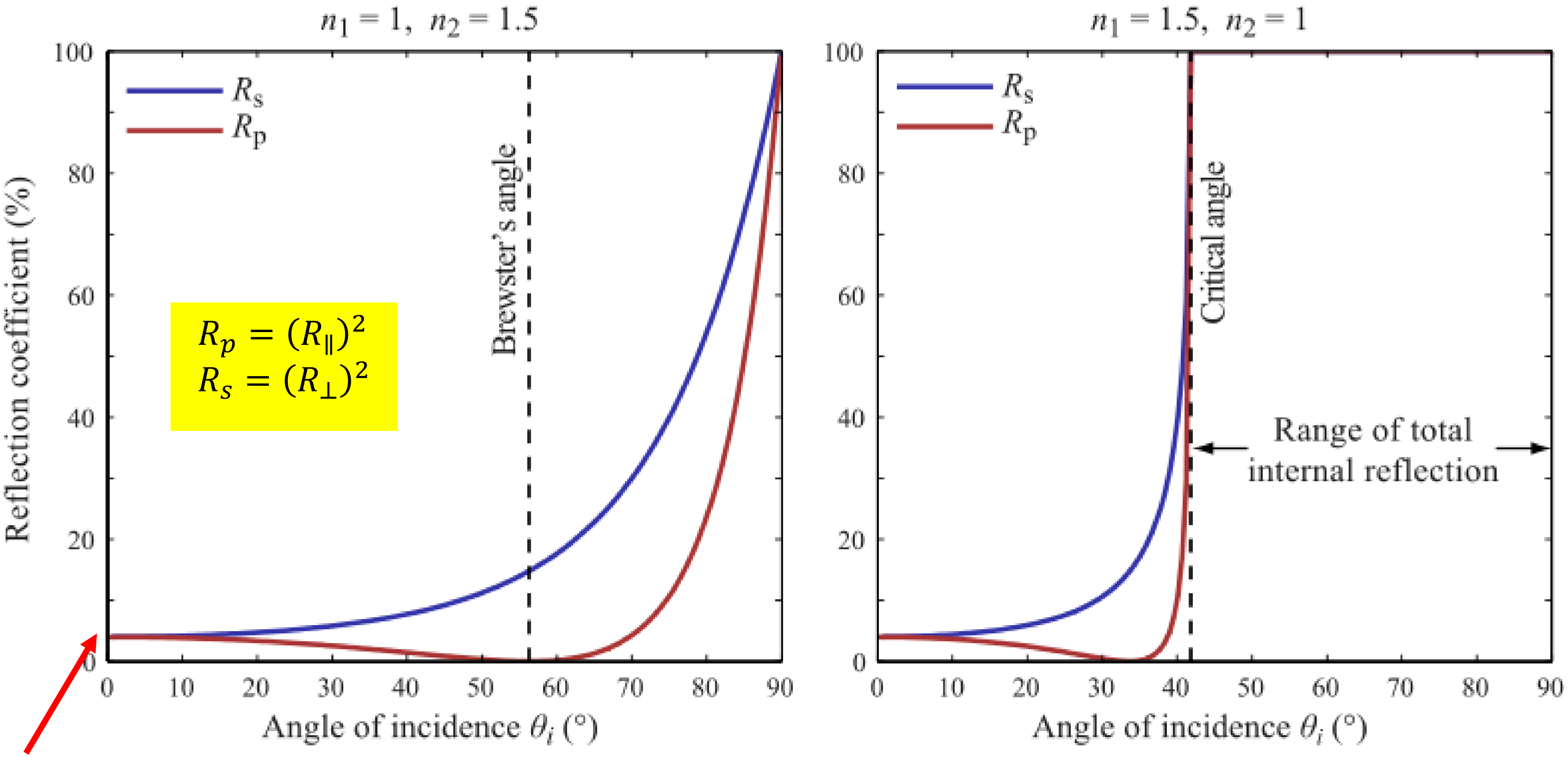
$As\ \tan(\theta + \phi) \to \infty, R_{\parallel} \to 0.$

This implies that $(\theta + \phi)$ = 90^0

By using the Snell's law, the Brewster angle can be found from

$$\tan\theta_B = {}^{n_2}/_{n_1}$$

External and internal reflections at air-glass interface



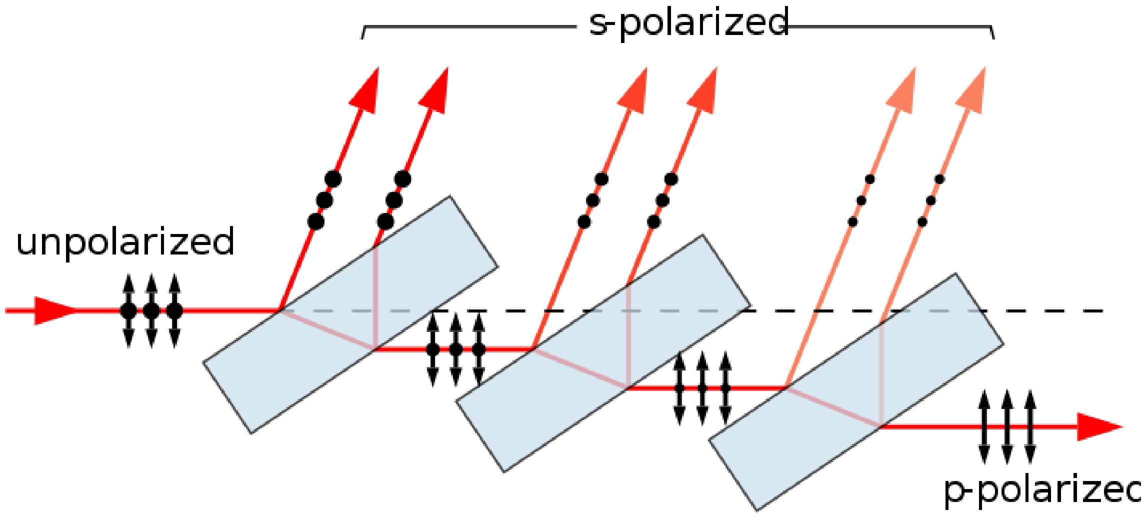
$$R_p = (R_{\parallel})^2$$
$$R_s = (R_{\perp})^2$$

$$R^2_{\theta \to 0} = \frac{I_r}{I_i} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \quad \text{x100}$$

~4% at an air-glass interface.

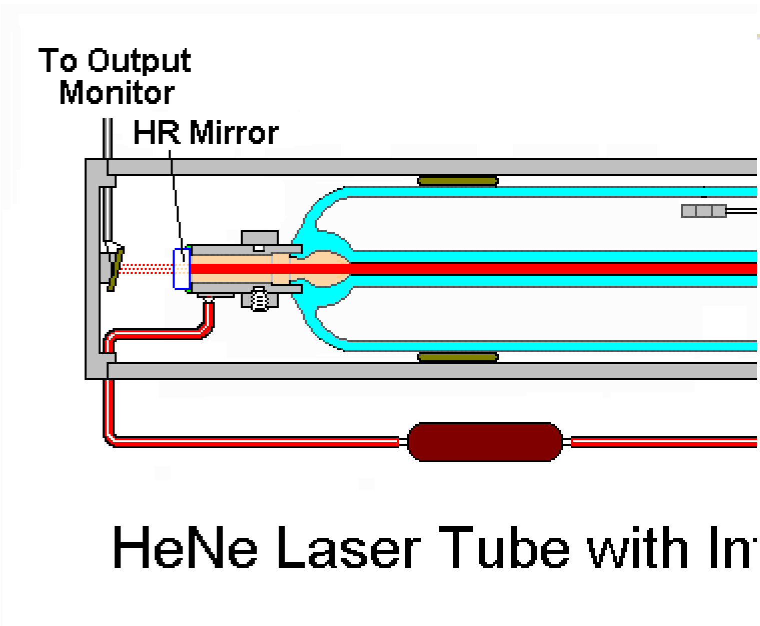
https://en.wikipedia.org/wiki/Fresnel_equations

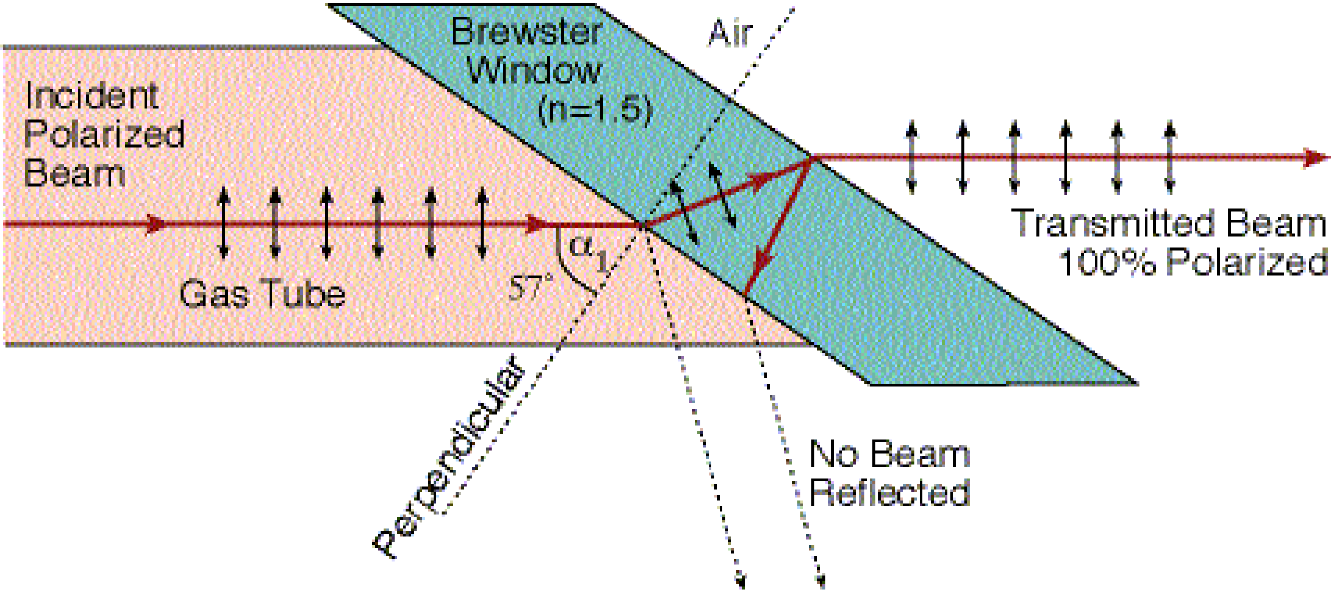
Application of Brewster angle



https://en.wikipedia.org/wiki/Polarizer

A stack of plates at Brewster's angle to a beam reflects off a fraction of the *s*-polarized light at each surface, leaving a *p*-polarized beam. Full polarization at Brewster's angle requires many more plates than shown. The arrows indicate the direction of the electrical field, not the magnetic field, which is perpendicular to the electric field.





https://perg.phys.ksu.edu/vqm/laserweb/Ch-7/F7s5t1p6.htm

S or P linear state of the output laser beam?

https://www.tau.ac.il/~phchlab/experiments_new/SemB04_Sucrose/02TheoreticalBackground.html